

Chapter 2 Affine Varieties.

$k = \bar{k}$ fixed algebraically closed field.

- all rings & fields containing k
- all ring homomorphisms preserve k .

(affine) variety := irreducible affine algebraic set in $\mathbb{A}^n = \mathbb{A}^n(k)$

§2.1 Coordinate Rings.

$V \subseteq \mathbb{A}^n$ nonempty variety. $\Rightarrow I(V) \triangleleft k[x_1, \dots, x_n]$ prime
 $\Rightarrow \Gamma(V) := k[x_1, \dots, x_n] / I(V) = \text{domain}$.

Def we called $\Gamma(V)$ the coordinate ring of V .

$\mathcal{F}(V, k) = \{ \text{functions from } V \text{ to } k \}$.

\supseteq ring str. containing k . (constant functions)

$\forall F \in k[x_1, \dots, x_n]$, define function:

$V \rightarrow k$ \longleftarrow polynomial function

$p \mapsto F(p)$

lem $\Gamma(V) \hookrightarrow \mathcal{F}(V, k)$ injective ring hom.

two ways to view elements in $\Gamma(V)$:
① function on V
② equivalence class of polynomials.

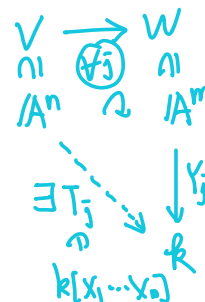
§ 2.2. Polynomial maps

$V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ varieties.

Def A mapping $\varphi: V \rightarrow W$ is called a **polynomial map**

if $\exists T_1, \dots, T_m \in k[x_1, \dots, x_n]$ s.t. $(\forall a_1, \dots, a_n)$

$$\varphi(a_1, \dots, a_n) = (T_1(a_1, \dots, a_n), \dots, T_m(a_1, \dots, a_n))$$



$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \tilde{\varphi}(f) := f \circ \varphi \searrow & & \downarrow f \\ & & k \end{array} \Rightarrow \begin{array}{ccc} \tilde{\varphi}: \mathcal{F}(W, k) & \rightarrow & \mathcal{F}(V, k) \\ \cup & & \cup \\ \Gamma(W) & & \Gamma(V) \end{array}$$

Fact: 1) If φ is polynomial map, then $\tilde{\varphi}(\Gamma(W)) \subset \Gamma(V)$

2) $f = F \bmod I(W) \Rightarrow \tilde{\varphi}(f) = F(T_1, \dots, T_m) \bmod I(V)$

$$\begin{aligned} \tilde{\varphi}(f) &= \tilde{\varphi}(F \bmod I(W)) \Big|_{(a_1, \dots, a_n)} = F \circ \varphi(a_1, \dots, a_n) \\ &= F(T_1(a_1, \dots, a_n), \dots, T_m(a_1, \dots, a_n)) \\ &= F(T_1, \dots, T_m) \Big|_{(a_1, \dots, a_n)} \end{aligned}$$

② $\Rightarrow \tilde{\varphi}(F \bmod I(W)) = F(T_1, \dots, T_m) \bmod I(V).$

$\forall T_1, \dots, T_m \in k[x_1, \dots, x_n] \Rightarrow$ polynomial map $T: \mathbb{A}^n \rightarrow \mathbb{A}^m$
 denote: $T = (T_1, \dots, T_m)$.

Prop $V \subseteq \mathbb{A}^n$ & $W \subseteq \mathbb{A}^m$

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$$1) \{ \varphi: V \rightarrow W \mid \text{polynomial map} \} \xleftrightarrow{|\cdot|} \{ \tilde{\varphi}: \Gamma(W) \rightarrow \Gamma(V) \mid \text{homomorphism} \}$$

2) any poly. map $\varphi: V \rightarrow W$ is a restriction of some poly. map $T: \mathbb{A}^n \rightarrow \mathbb{A}^m$.

Pf: $\forall \alpha: \Gamma(W) \rightarrow \Gamma(V) \quad \alpha(x_i \bmod I(W))$

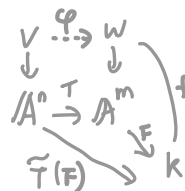
$$\begin{array}{ccc} \pi_W \uparrow & \uparrow \pi_V & \uparrow \\ k[x_1, \dots, x_m] & k[x_1, \dots, x_n] & \Gamma \\ & & T_i \end{array}$$

choose $T_i \in k[x_1, \dots, x_n]$ s.t. $T_i \bmod I(V) = \alpha(x_i \bmod I(W))$

$\Rightarrow T = (T_1, \dots, T_m): \mathbb{A}^n \rightarrow \mathbb{A}^m$ poly. map

$\Rightarrow \tilde{T}: \Gamma(\mathbb{A}^m) = k[x_1, \dots, x_m] \rightarrow \Gamma(\mathbb{A}^n) = k[x_1, \dots, x_n]$

$$\pi_V \circ \tilde{T}(x_i) = \alpha \circ \pi_W(x_i)$$



$$\begin{array}{ccc} \Gamma(W) & \xrightarrow{\alpha} & \Gamma(V) \\ \pi_W \uparrow & & \uparrow \pi_V \\ k[x_1, \dots, x_m] & \xrightarrow{\tilde{T}} & k[x_1, \dots, x_n] \end{array} \Rightarrow \tilde{T}(I(W)) \subseteq I(V) \Leftrightarrow T(V) \subseteq W$$

$$\begin{aligned} & T(v) \in W \quad \forall v \in V \\ \Leftrightarrow & F(T(v)) = 0 \quad \forall F \in I(W), \forall v \in V \\ \Leftrightarrow & \tilde{T}(F)(v) = 0 \quad \forall F \in I(W), \forall v \in V \\ \Leftrightarrow & \tilde{T}(F) \in I(V) \end{aligned}$$

$\Rightarrow T$ restricts to a polynomial map $\varphi: V \rightarrow W$.

• $\hat{\varphi} = \alpha$.

$$\begin{aligned} \forall v \in V, \tilde{\varphi}(f)(v) &= f \circ \varphi(v) = F \circ T(v) = \tilde{T}(F)(v) \\ &= \pi_V(\tilde{T}(F))(v) = \alpha(\pi_W(F))(v) = \alpha(f)(v) \end{aligned}$$

• bijection: $V \begin{array}{c} \xrightarrow{\varphi_1} \\ \xrightarrow{\varphi_2} \end{array} W \quad \varphi_1 \neq \varphi_2 \Rightarrow \exists v \in V \text{ s.t. } \varphi_1(v) \neq \varphi_2(v)$

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$\Rightarrow \exists f \in \Gamma(W)$ s.t.
 $f(\varphi_1(v)) \neq f(\varphi_2(v))$
 $\Rightarrow \tilde{\varphi}_1 \neq \tilde{\varphi}_2$

Def A polynomial map $\varphi: V \rightarrow W$ is an isomorphism, if \exists poly. map

$$\psi: W \rightarrow V \text{ s.t. } \psi \circ \varphi = \text{id} \ \& \ \varphi \circ \psi = \text{id}.$$

Fact: $V \cong W \Leftrightarrow \Gamma(V) \cong \Gamma(W)$.