Chapter 2 Affine Varieties.

$k = \bar{k}$ fixed algebraically closed field.

- all rings & fields containg k
- all ring homomorphisms preserve k.

(affine) variety := irreducible affine algebraic set in $\mathbb{A}^n = \mathbb{A}^n(k)$

§2.1 Coordinate Rings.

$V \subseteq \mathbb{A}^n$ nonempty. variety. $\Rightarrow I(V) \triangleleft k[x_1, \cdots, x_n]$ prime

$\Rightarrow \Gamma(V) := k[x_1, \cdots, x_n] / I(V)$ = domain.

Def We called $\Gamma(V)$ the coordinate ring of V.

$\mathcal{F}(V, k) = \{$ functions from V to $k\}$.

$\uparrow$ ring str. containing k. (constant functions)

$\forall F \in k[x_1, \cdots, x_n]$, define function:

$$V \to k \qquad \leftarrow \text{polynomial function}$$
$$P \mapsto F(P)$$

Lem $\Gamma(V) \hookrightarrow \mathcal{F}(V, k)$ injective ring hom.

two ways to view elements in $\Gamma(V)$: ① function on V
② equivalence class of polynomials.

§ 2.2. Polynomial maps

$V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ varieties.

Def A mapping $\varphi : V \to W$ is called a **polynomial map** if $\exists T_1, \cdots, T_m \in k[X_1, \cdots, X_n]$ s.t. $(\forall a_1, \cdots, a_n)$

$$\varphi(a_1, \cdots, a_n) = \left(T_1(a_1, \cdots, a_n), \cdots, T_m(a_1, \cdots, a_n)\right)$$

$$\begin{tikzcd} V \arrow[r, "\forall j"] \arrow[d, phantom, "\cap"] & W \arrow[d, phantom, "\cap"] \\ \mathbb{A}^n \arrow[dr, dashed, "\exists T_j \in k[X_1 \cdots X_n]"'] & \mathbb{A}^m \arrow[d, "y_j"] \\ & k \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\varphi", "(T_1, \cdots, T_m)"'] \arrow[dr, "\tilde{\varphi}(f) := f \circ \varphi"'] & W \arrow[d, "f", "F"'] \\ & k \end{tikzcd} \quad \Rightarrow \quad \tilde{\varphi} : \mathcal{F}(W, k) \to \mathcal{F}(V, k)$$

$$\begin{matrix} \mathcal{F}(W,k) & & \mathcal{F}(V,k) \\ \cup & & \cup \\ \Gamma(W) & & \Gamma(V) \end{matrix}$$

Fact:
1) If φ is polynomial map, then $\tilde{\varphi}(\Gamma(W)) \subset \Gamma(V)$
2) $f = F \bmod I(W) \Rightarrow \tilde{\varphi}(f) = F(T_1, \cdots, T_m) \bmod I(V)$

Pf:
$$\tilde{\varphi}(F \bmod I(W))\Big|_{(a_1, \cdots, a_n)} = F \circ \varphi(a_1, \cdots, a_n)$$
$$= F\left(T_1(a_1, \cdots, a_n), \cdots, T_m(a_1 \cdots a_n)\right)$$
$$= F(T_1, \cdots, T_m)\Big|_{(a_1 \cdots a_n)}$$

2 $\Rightarrow \tilde{\varphi}(F \bmod I(W)) = F(T_1, \cdots, T_m) \bmod I(V)$.

$\forall\, T_1, \cdots, T_m \in k[X_1, \dots, X_n] \Rightarrow$ polynomial map $T : \mathbb{A}^n \longrightarrow \mathbb{A}^m$

denote : $T = (T_1, \cdots, T_n)$.

Prop $V \subseteq \mathbb{A}^n$ & $W \subseteq \mathbb{A}^m$

1) $\underbrace{\{\varphi : V \to W \mid \text{polynomial map}\}}^{\text{代数几何}} \xleftrightarrow{1:1} \overbrace{\{\tilde{\varphi} : \Gamma(W) \to \Gamma(V) \mid \text{homomorphism}\}}^{\text{交换代数}}$

2) any poly. map $\varphi : V \to W$ is a restriction of some poly. map $T : \mathbb{A}^n \to \mathbb{A}^m$.

Pf: $\forall$
$$\begin{tikzcd} \Gamma(W) \arrow[r, "\alpha"] & \Gamma(V) & \alpha(X_i \bmod I(W)) \\ k[X_1, \cdots, X_m] \arrow[u, "\pi_W"] & k[X_1, \cdots, X_n] \arrow[u, "\pi_V"'] & T_i \arrow[u, mapsto] \end{tikzcd}$$

choose $T_i \in k[X_1, \cdots, X_n]$ s.t. $T_i \bmod I(V) = \alpha(X_i \bmod I(W))$

$\Rightarrow T = (T_1, \cdots, T_n) : \mathbb{A}^n \to \mathbb{A}^m$ poly. map

$\Rightarrow \tilde{T} : \Gamma(\mathbb{A}^m) = k[X_1, \cdots, X_m] \to \Gamma(\mathbb{A}^n) = k[X_1, \cdots, X_n]$

$$\pi_V \circ \tilde{T}(X_i) = \alpha \circ \pi_W(X_i)$$

$$\begin{tikzcd} V \arrow[r, dashed, "\varphi"] \arrow[d] & W \arrow[d] \arrow[ddr, bend left, "f"] & \\ \mathbb{A}^n \arrow[r, "T"] \arrow[drr, bend right, "\tilde{T}(F)"'] & \mathbb{A}^m \arrow[dr, "F"] & \\ & & k \end{tikzcd}$$

$$\begin{tikzcd} \Gamma(W) \arrow[r, "\alpha"] & \Gamma(V) \\ k[X_1, \cdots, X_m] \arrow[u, "\pi_W"] \arrow[r, "\tilde{T}"] & k[X_1, \cdots, X_n] \arrow[u, "\pi_V"'] \\ I(W) \arrow[u, hook] \arrow[r, dashed] & I(V) \arrow[u, hook] \end{tikzcd}$$

$\Rightarrow \tilde{T}(I(W)) \subseteq I(V) \Leftrightarrow T(V) \subseteq W$

$\Updownarrow$

$T(v) \in W \quad \forall v \in V$

$\Leftrightarrow F(T(v)) = 0 \quad \forall F \in I(W), \forall v \in V$

$\Leftrightarrow \tilde{T}(F)(v) = 0 \quad \forall F \in I(W) \quad \forall v \in V$

$\Leftrightarrow \tilde{T}(F) \in I(V)$

$\Rightarrow$ T restricts to a polynomial map $\varphi : V \to W$.

- $\tilde{\varphi} = a$.

$\forall\, v \in V$, $\tilde{\varphi}(f)(v) = f \circ \varphi(v) = F \circ T(v) = \tilde{T}(F)\,(v)$

$= \pi_V(\tilde{T}(F))\,(v) = \alpha\,(\pi_W(F))\,(v) = \alpha(f)(v)$

- bijection: $\begin{tikzcd} V \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] \arrow[dr] & W \arrow[d] \\ & k \end{tikzcd}$

$\varphi_1 \neq \varphi_2 \Rightarrow \exists\, v \in V$ s.t. $\varphi_1(v) \neq \varphi_2(v)$

$\Rightarrow \exists\, f \in \Gamma(W)$ s.t.

$f(\varphi_1(v)) \neq f(\varphi_2(v))$

$\Rightarrow \tilde{\varphi}_1 \neq \tilde{\varphi}_2$

Def A polynomial map $\varphi : V \to W$ is an *isomorphism*, if $\exists$ poly. map $\psi : W \to V$ s.t. $\psi \circ \varphi = \mathrm{id}$ & $\varphi \circ \psi = \mathrm{id}$.

Fact: $V \cong W \iff \Gamma(V) \cong \Gamma(W)$.